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# The set of prime divisors of generalized denominator ideals

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**Abstract** Let $R$ be a Noetherian domain with quotient field $K$. Let $A$ be an integral domain which contains $R$ and whose elements are algebraic over $K$. We define $\mathrm{Eass}_R(A/R)$ to be the set of prime ideals $\mathfrak{p}$'s of $R$ such that $\mathfrak{p}$ is a prime divisor of a generalized denominator ideal $I_{[\beta]}$ for some $\beta \in A$. Assume that $A = R[\alpha_1, \ldots, \alpha_n]$. We investigate the relation between $\mathrm{Eass}_R(A/R)$ and $\cup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$. Furthermore, for a finite subset $\Delta$ of $\mathrm{Eass}_R(A/R)$, we construct the subring $A_\Delta$ of $A$ such that $A_\Delta$ is the largest among those $B$'s with $\Delta = \mathrm{Eass}_R(B/R)$.

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**الملخص**

ليكن $R$ مجالاً نوثيرياً حقل كسوره $K$. ليكن $A$ مجالاً صحيحاً يحتوي $R$ وعناصره جبرية على $K$. نعرف $\mathrm{Eass}_R(A/R)$ على أنها مجموعة المثاليات p من $R$ حيث p قاسم أولي لمثالي المقام المعمَّم $I_\beta$ وذلك لأحد العناصر $\beta \in A$. افترض أن $A = R[\alpha_1, \cdots, \alpha_n]$. نبحث العلاقة بين $\mathrm{Eass}_R(A/R)$ وَ $\cup_{i=1}^{n} Ass_R(R/I_{[\alpha_i]})$. بالإضافة إلى ذلك، نركب الحلقة الجزئية $A_\Delta$ من $A$ حيث $\Delta$ مجموعة جزئية منتهية من $\mathrm{Eass}_R(A/R)$ وَ $A_\Delta$ أكبر المجالات $B$ التي تحقق $\Delta = \mathrm{Eass}_R(A/R)$.

## 1 Introduction

Let $R$ be a Noetherian domain with quotient field $K$. An anti-integral extension is the simplest case of algebraic simple extensions. Let $\alpha$ be an element which is algebraic over $K$. Considering an anti-integral extension $R[\alpha]$ over $R$ in the case $\alpha$ is in $K$, we have only to treat the usual denominator ideal $I_\alpha$. To treat a higher degree case of $\alpha$, we need the generalized denominator ideal $I_{[\alpha]}$.

Let $A$ be an integral domain containing $R$ and assume that $A = R[\alpha_1, \ldots, \alpha_n]$ where $\alpha_1, \ldots, \alpha_n$ are algebraic elements over $K$. We investigate the set of prime ideals $\mathfrak{p}$'s of $R$ such that $\mathfrak{p}$ is a prime divisor of a

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generalized denominator ideal $I_{[\beta]}$ for some $\beta \in A$ and denote the set by $\mathrm{Eass}_R(A/R)$. If $R$ is normal, then we have

$$\mathrm{Eass}_R(A/R) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}).$$

This means that all prime divisors of $I_{[\beta]}$ for any $\beta$ in $A$ are elements of $\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$. Furthermore, if $A = R[\alpha_1, \dots, \alpha_n] = R[\beta_1, \dots, \beta_m]$, then

$$\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) = \mathrm{Eass}_R(A/R) = \bigcup_{j=1}^{m} \mathrm{Ass}_R(R/I_{[\beta_j]}).$$

In Sect. 2, we investigate the relation between $\mathrm{Eass}_R(A/R)$ and the generalized denominator ideals of generators $\alpha_1, \dots, \alpha_n$ under some conditions (Theorems 2.7, 2.8 and 2.9).

In Sect. 3, we construct a subring $A_\Delta$ of $A$ by making use of a finite subset $\Delta$ of $\mathrm{Eass}_R(A/R)$ and we prove an interesting property of $\Delta = \mathrm{Eass}_R(A_\Delta/R)$ (Theorem 3.4).

## 2 Associated primes

Let $R$ be a Noetherian domain with quotient field $K$ and $R[X]$ a polynomial ring over $R$ in an indeterminate $X$. Let $\alpha$ be an element of an algebraic field extension of $K$ and let $\pi : R[X] \longrightarrow R[\alpha]$ be the $R$-algebra homomorphism induced by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of $\alpha$ over $K$ with $\deg \varphi_\alpha = d$, and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \dots + \eta_d, \ (\eta_1, \dots, \eta_d \in K).$$

We define the generalized denominator ideal $I_{[\alpha]}$ of $\alpha$ by $I_{[\alpha]} = \bigcap_{i=1}^{d}(R :_R \eta_i)$ where $(R :_R \eta_i) = \{c \in R; c\eta_i \in R\}$. Note that $I_{[\alpha]} \neq (0)$ because $\alpha$ is algebraic over $K$. An element $\alpha$ is called an anti-integral element of degree $d$ over $R$ if $\mathrm{Ker}\,\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. We set $\mathrm{Dp}_1(R) = \{\mathfrak{p} \in \mathrm{Spec}\,R; \mathrm{depth}\,R_\mathfrak{p} = 1\}$ and $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$ where $(1, \eta_1, \dots, \eta_d)$ is the $R$-module generated by $1, \eta_1, \dots, \eta_d$. The element $\alpha$ is said to be a super-primitive element of degree $d$ over $R$ if $J_{[\alpha]} \not\subset \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Dp}_1(R)$.

Let $A$ be an integral domain containing $R$. Assume that every element of $A$ is algebraic over $K$. We define $\mathrm{Eass}_R(A/R)$ by

$$\mathrm{Eass}_R(A/R) = \{\mathfrak{p} \in \mathrm{Spec}\,R; \mathfrak{p} \text{ is a prime divisor of } I_{[\beta]} \text{ for some } \beta \in A\}.$$

Since every prime ideal in $\mathrm{Eass}_R(A/R)$ is a prime divisor of a divisorial ideal and each prime divisor of a divisorial ideal is of depth one by [7, Proposition 1.10], we see that $\mathrm{Eass}_R(A/R) \subset \mathrm{Dp}_1(R)$.

Our notation is standard and our general reference for unexplained terms is [5].

**Lemma 2.1** [6, Theorem 2.2] *Let $R$ be a Noetherian domain and $\alpha$ an anti-integral element over $R$. Set $A = R[\alpha]$. Then, the following two conditions are equivalent:*

(i) $I_{[\alpha]} = R$.
(ii) *$A$ is an integral extension of $R$.*

**Lemma 2.2** [2, Lemma 1.1 (1)] *Let $R$ be an integral domain and $\alpha$ an anti-integral element of degree $d$ over $R$. Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \dots + \eta_d$ be the monic minimal polynomial of $\alpha$ over the quotient field of $R$. Let $\mathfrak{p}$ be an element of* $\mathrm{Spec}\,R$. *Then, the following two assertions hold:*

(1) *If $\alpha$ is an anti-integral element of degree $d$ over $R$, then $\alpha$ is also an anti-integral element of degree $d$ over $R_\mathfrak{p}$.*
(2) $I_{R_\mathfrak{p},[\alpha]} = I_{[\alpha]}R_\mathfrak{p}$ *where* $I_{R_\mathfrak{p},[\alpha]} = \bigcap_{i=1}^{d}(R_\mathfrak{p} :_{R_\mathfrak{p}} \eta_i)$.

**Proposition 2.3** *Let $R$ be a Noetherian domain and $\alpha$ a super-primitive element over $R$. Let $\mathfrak{p}$ be an element of* $\mathrm{Dp}_1(R)$. *If $\mathfrak{p} \supset I_{[\alpha]}$, then $\mathfrak{p}$ is a prime divisor of $I_{[\alpha]}$.*

*Proof* Since $\alpha$ is a super-primitive element over $R$ and $\mathfrak{p}$ is in $\mathrm{Dp}_1(R)$, we get $J_{[\alpha]}R_{\mathfrak{p}} = R_{\mathfrak{p}}$. Hence $I_{[\alpha]}R_{\mathfrak{p}}$ is an invertible ideal of $R_{\mathfrak{p}}$. This implies that $I_{[\alpha]}R_{\mathfrak{p}}$ is a principal ideal of $R_{\mathfrak{p}}$ by [4, Theorem 59]. Hence there exists a non-zero element $a$ of $I_{[\alpha]}$ such that $I_{[\alpha]}R_{\mathfrak{p}} = aR_{\mathfrak{p}}$. By [7, Proposition 1.10, (i) $\Rightarrow$ (ii)], $\mathfrak{p}R_{\mathfrak{p}}$ is a prime divisor of $aR_{\mathfrak{p}} = I_{[\alpha]}R_{\mathfrak{p}}$ because $\mathfrak{p}R_{\mathfrak{p}} \supset aR_{\mathfrak{p}}$ and depth $\mathfrak{p}R_{\mathfrak{p}} = 1$. Therefore, $\mathfrak{p}$ is a prime divisor of $I_{[\alpha]}$ by [3, Chap. 4, §1, Proposition 5]. □

Let $R$ be a Noetherian domain with quotient field $K$ and let $\alpha$ be an algebraic element over $K$. Let $\mathfrak{p}$ be a prime ideal of $R$ and consider the following two conditions:

(i) $\mathfrak{p}$ is a prime divisor of $I_{[\alpha]}$.
(ii) $\mathfrak{p} \in \mathrm{Dp}_1(R)$ and $\mathfrak{p} \supset I_{[\alpha]}$.
Then, the implication (i) $\Longrightarrow$ (ii) holds by [7, Proposition 1.10] and we know that the implication (ii) $\Longrightarrow$ (i) does not hold by the following example. Proposition 2.3 shows that the implication (ii) $\Longrightarrow$ (i) holds under the additional assumption that $\alpha$ is a super-primitive element over $R$.

*Example 2.4* Let $k$ be a field and let $s, t$ be indeterminates over $k$. Set $R = k[s, st, t^2, t^3]$.

(1) Let $f$ be an element of $k[s, t]$ and write

$$f = f_0(s) + f_1(s)t + f_2(s)t^2 + \cdots, \; (f_0(s), f_1(s), f_2(s), \ldots \in k[s]).$$

Then, $f$ is in $R$ if and only if $f_1(0) = 0$.
(2) Let $\mathfrak{p}$ be an ideal of $R$ generated by $s, st, t^2$ and $t^3$. Let $\mathfrak{q}$ be an ideal of $R$ generated by $st, t^2$ and $t^3$. Then, both $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of $R$.
(3) Set $\alpha = s/t$. Then, $I_{[\alpha]} = \mathfrak{q}$.
(4) Though depth$R_{\mathfrak{p}} = 1$ and $\mathfrak{p} \supset I_{[\alpha]}$, $\mathfrak{p}$ is not a prime divisor of $I_{[\alpha]}$.
(5) $\alpha$ is not a super-primitive element over $R$.

*Proof* (1) Let $i$ be a positive integer with $i \geqq 2$. Then, $f_i(s)t^i$ is in $R$. Furthermore, $f_0(s)$ is in $R$. Hence $f$ is in $R$ if and only if $f_1(s)t$ is in $R$. Moreover, $f_1(s)t$ is in $R$ if and only if $f_1(0) = 0$. Hence we prove the assertion (1).

(2) First, we shall show that $R$ is isomorphic to $k[X, Y, Z, W]/(X^2Z - Y^2, Z^3 - W^2, XW - YZ, YW - XZ^2)$. Let $\phi : k[X, Y, Z, W] \longrightarrow R$ be the $k$-algebra homomorphism induced by $\phi(X) = s$, $\phi(Y) = st$, $\phi(Z) = t^2$, $\phi(W) = t^3$. Let $\mathfrak{P}$ be the ideal generated by $X^2Z - Y^2$, $Z^3 - W^2$, $XW - YZ$, $YW - XZ^2$. The inclusion $\mathfrak{P} \subset \mathrm{Ker}\,\phi$ is obvious. We will prove the converse inclusion. Let $f$ be an arbitrary element of $\mathrm{Ker}\,\phi$. Then, it is easily verified that $f$ is written as

$$\begin{aligned} f = &\, a_0(X, Z) + a_1(X, Z)W + (a_2(X, Z) + a_3(X, Z)W)Y \\ &+ g(X, Z, W)(X^2Z - Y^2) + h(X, Z)(Z^3 - W^2). \end{aligned}$$

Then, we have

$$a_0(s, t^2) + a_1(s, t^2)t^3 + a_2(s, t^2)st + a_3(s, t^2)st^4 = 0.$$

Separating even degree terms from odd degree terms with respect to $t$, we get

$$a_0(s, t^2) + a_3(s, t^2)st^4 = 0, \;\; a_1(s, t^2)t^3 + a_2(s, t^2)st = 0.$$

Since $s$ and $t^2$ are algebraically independent over $k$, we see that

$$a_0(X, Z) + a_3(X, Z)XZ^2 = 0, \;\; a_1(X, Z)Z + a_2(X, Z)X = 0.$$

Then, there exists an element $b(X, Z)$ of $k[X, Z]$ such that $a_1(X, Z) = b(X, Z)X$ and $a_2(X, Z) = -b(X, Z)Z$. Therefore,

$$\begin{aligned} f = &\, a_3(X, Z)(YW - XZ^2) + b(X, Z)(XW - YZ) \\ &+ g(X, Z, W)(X^2Z - Y^2) + h(X, Z)(Z^3 - W^2). \end{aligned}$$

Hence $f$ is in $\mathfrak{P}$, and $\mathfrak{P} = \mathrm{Ker}\,\phi$. This implies the required result.

Since $R$ is isomorphic to

$$k[X, Y, Z, W]/(X^2Z - Y^2, Z^3 - W^2, XW - YZ, YW - XZ^2),$$

we get $R/\mathfrak{p}$ is isomorphic to

$$\begin{aligned} &k[X, Y, Z, W]/(X^2Z - Y^2, Z^3 - W^2, XW - YZ, YW - XZ^2, X, Y, Z, W) \\ &\cong k[X, Y, Z, W]/(X, Y, Z, W) \cong k. \end{aligned}$$

and

$$R/\mathfrak{q} \cong k[X, Y, Z, W]/(Y, Z, W) \cong k[X].$$

Hence both $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of $R$.

(3) Since $\alpha = s/t = st/t^2$, we have $\alpha \in K$. Then, $\varphi_\alpha(X) = X - s/t$. The inclusion $\mathfrak{q} \subset I_{[\alpha]}$ is obvious. We shall prove the converse inclusion. Let $f$ be an arbitrary element of $I_{[\alpha]}$. Then, $fs/t$ is in $R$. Write

$$f = f_0(s) + f_1(s)t + f_2(s)t^2 + \cdots, (f_0(s), f_1(s), f_2(s), \ldots \in k[s]).$$

Then, $f_2(s)t^2 + \cdots$ is in $\mathfrak{q}$. On the other hand, $f_1(0) = 0$ because $f$ is in $R$. Hence $f_1(s)t \subset (st) \subset \mathfrak{q}$. Furthermore, $f_0(s)s/t$ is in $R$ because $(f - f_0(s))s/t$ and $fs/t$ are in $R$. This implies that $f_0(s) = 0$ and $f$ is in $\mathfrak{q}$. Therefore, $I_{[\alpha]} \subset \mathfrak{q}$, and $I_{[\alpha]} = \mathfrak{q}$.

(4) We will show that $s$ is a maximal $R$-sequence. Note that $s$ is a non-zero divisor of $R$ and $st \notin (s)$ because $t \notin R$. Furthermore, $\mathfrak{p}st = (s^2t, s^2t^2, st^3, st^4) \subset (s)$. Hence $s$ is a maximal $R$-sequence. Since every maximal $R$-sequence has the same length, we see that $\operatorname{depth} R_\mathfrak{p} = 1$. It is obvious that $I_{[\alpha]} = \mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{p}$ is not a prime divisor of $I_{[\alpha]}$.

(5) Since $I_{[\alpha]} = \mathfrak{q} = (st, t^2, t^3)$, we have $J_{[\alpha]} = I_{[\alpha]}(1, -s/t) = (st, t^2, t^3, s^2) \subset \mathfrak{p}$. Hence $\alpha$ is not a super-primitive element over $R$ because $\operatorname{depth} R_\mathfrak{p} = 1$. □

*Remark 2.5* In Example 2.4, $\alpha$ is an anti-integral element over $R$.

*Proof* Let $\overline{R}$ be the integral closure of $R$ in $K$. Then, $\overline{R} = k[s, t]$. Let $\pi_0 : \overline{R}[X] \longrightarrow \overline{R}[\alpha]$ be the $\overline{R}$-algebra homomorphism induced by $\pi_0(X) = \alpha$. We have only to prove that $\operatorname{Ker} \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. We know that

$$I_{[\alpha]}\varphi_\alpha(X)R[X] = (stX - s^2, t^2X - st, t^3X - st^2)R[X].$$

The inclusion $\supset$ is obvious. We will show the converse inclusion. Let $F(X)$ be an arbitrary element of $\operatorname{Ker} \pi$. Then, $F(X)$ is in $\operatorname{Ker} \pi_0$. Hence there exists an element $G(X)$ of $Q(\overline{R})[X]$ such that $F(X) = (tX - s)G(X)$ where $Q(\overline{R})$ denotes the quotient field of $\overline{R}$. Since $\overline{R}$ is a UFD, we see that $G(X)$ is in $\overline{R}[X]$ by Gauss' lemma. Write

$$\begin{aligned} G(X) = g_0(s, X) + g_1(s, X)t + g_2(s, X)t^2 + \cdots, \\ (g_0(s, X), g_1(s, X), g_2(s, X), \ldots \in k[s, X]). \end{aligned}$$

Let $i$ be a positive integer. If $i$ is even, then we have $(tX - s)g_i(s, X)t^i = (t^3X - st^2)g_i(s, X)t^{i-2}$ and $g_i(s, X)t^{i-2} \in R[X]$. If $i$ is odd, then we get $(tX - s)g_i(s, X)t^i = (t^2X - st)g_i(s, X)t^{i-1}$ and $g_i(s, X)t^{i-1} \in R[X]$. Therefore, $\sum_{i \geqq 1}(tX - s)g_i(s, X)t^i$ is in $I_{[\alpha]}\varphi_\alpha(X)R[X]$. Furthermore, $(tX - s)g_0(s, X)$ is in $R[X]$ because $F(X)$ and $\sum_{i \geqq 1}(tX - s)g_i(s, X)t^i$ are in $R[X]$. Hence $g_0(0, X) = 0$ and it is easily seen that $(tX - s)g_0(s, X) = (stX - s^2)\overline{g}_0(s, X)$ is in $I_{[\alpha]}\varphi_\alpha(X)R[X]$ for some $\overline{g}_0(s, X)$ in $R[X]$. This implies that $F(X) \in I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence $\operatorname{Ker} \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. □

**Proposition 2.6** *Let $R$ be a Noetherian domain with quotient field $K$. Let $\alpha_1, \ldots, \alpha_n$ be algebraic elements over $K$ and set $A = R[\alpha_1, \ldots, \alpha_n]$. Assume that every element of $A$ is an anti-integral element over $R$. Let $\mathfrak{p}$ be an element of* $\operatorname{Dp}_1(R)$. *Then, $\mathfrak{p} \in \operatorname{Eass}_R(A/R)$ if and only if $\mathfrak{p} \supset I_{[\alpha_1]} \cap \cdots \cap I_{[\alpha_n]}$.*

*Proof* We shall prove the "only if" part. By the definition of $\mathrm{Eass}_R(A/R)$, there exists an element $\beta$ of $A$ such that $\mathfrak{p}$ is a prime divisor of $I_{[\beta]}$. Suppose that $\mathfrak{p} \not\supset I_{[\alpha_1]} \cap \cdots \cap I_{[\alpha_n]}$. Then, $\mathfrak{p} \not\supset I_{[\alpha_1]}, \ldots, \mathfrak{p} \not\supset I_{[\alpha_n]}$. By Lemmas 2.1 and 2.2, this implies that $\alpha_1, \ldots, \alpha_n$ are integral over $R_{\mathfrak{p}}$. Hence $A_{\mathfrak{p}}$ is an integral extension of $R_{\mathfrak{p}}$. So $\beta$ is integral over $R_{\mathfrak{p}}$. Lemmas 2.1 and 2.2 show that $I_{[\beta]} R_{\mathfrak{p}} = I_{R_{\mathfrak{p}},[\beta]} = R_{\mathfrak{p}}$. This contradicts the fact $\mathfrak{p} \supset I_{[\beta]}$.

We will prove the "if" part. If $\mathfrak{p} \supset I_{[\alpha_1]} \cap \cdots \cap I_{[\alpha_n]}$, then there exists an index $i$ such that $\mathfrak{p} \supset I_{[\alpha_i]}$. This means that $\mathfrak{p}$ is in $\mathrm{Eass}_R(A/R)$. □

**Theorem 2.7** *Let $R$ be a Noetherian domain and $\alpha_1, \ldots, \alpha_n$ super-primitive elements over $R$. Set $A = R[\alpha_1, \ldots, \alpha_n]$ and assume that every element of $A$ is an anti-integral element over $R$. Then, the following holds:*

$$\mathrm{Eass}_R(A/R) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}).$$

*Proof* The inclusion $\mathrm{Eass}_R(A/R) \supset \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$ is clear from the definition of $\mathrm{Eass}_R(A/R)$. We will prove the converse inclusion. Let $\mathfrak{p}$ be an arbitrary element of $\mathrm{Eass}_R(A/R)$. Then by Proposition 2.6, we have $\mathfrak{p} \supset I_{[\alpha_1]} \cap \cdots \cap I_{[\alpha_n]}$. Hence there exists an index $j$ such that $\mathfrak{p} \supset I_{[\alpha_j]}$. Note that $\mathfrak{p}$ is in $\mathrm{Dp}_1(R)$. Then, Proposition 2.3 asserts that $\mathfrak{p}$ is a prime divisor of $I_{[\alpha_j]}$. This shows that $\mathfrak{p}$ is in $\mathrm{Ass}_R(R/I_{[\alpha_j]})$, hence $\mathfrak{p}$ is in $\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$. □

**Theorem 2.8** *Let $R$ be a Noetherian normal domain with quotient field $K$. Let $\alpha_1, \ldots, \alpha_n$ be algebraic elements over $K$ and set $A = R[\alpha_1, \ldots, \alpha_n]$. Then, the following holds:*

$$\mathrm{Eass}_R(A/R) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}).$$

*Proof* Note that Noetherian normal domains are Krull domains. By [6, Theorem 1.13], we see that $\alpha_1, \ldots, \alpha_n$ are super-primitive elements over $R$. Furthermore, every element of $A$ is an algebraic over $K$, and so it is a super-primitive element over $R$. Super-primitive elements are anti-integral elements by [6, Theorem 1.12]. Hence by Theorem 2.7, we get the required result. □

Let $\overline{R}$ be the integral closure of $R$ in the quotient field $K$. Let $\mathfrak{c}(\overline{R}/R)$ be the conductor ideal of $R$ in $\overline{R}$, that is, $\mathfrak{c}(\overline{R}/R) = \{a \in R;\, a\overline{R} \subset R\}$.

We would like to generalize Theorem 2.7 by deleting the assumption that every element of $A$ is an anti-integral element over $R$.

**Theorem 2.9** *Let $R$ be a Noetherian domain with quotient field $K$. Let $\overline{R}$ be the integral closure of $R$ in $K$. Let $\alpha_1, \ldots, \alpha_n$ be algebraic elements over $K$ and set $A = R[\alpha_1, \ldots, \alpha_n]$. Then, the following assertion holds:*

$$\mathrm{Eass}_R(A/R) \subset \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \cup \{\mathfrak{p} \in \mathrm{Dp}_1(R);\, \mathfrak{p} \supset \mathfrak{c}(\overline{R}/R)\}.$$

*Proof* Let $\mathfrak{p}$ be an arbitrary element of $\mathrm{Eass}_R(A/R)$. Then, $\mathfrak{p}$ is in $\mathrm{Dp}_1(R)$. So we have only to prove that, if $\mathfrak{p} \not\supset \mathfrak{c}(\overline{R}/R)$, then $\mathfrak{p}$ is in $\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$. Assume that $\mathfrak{p} \not\supset \mathfrak{c}(\overline{R}/R)$. Then, we have $R_{\mathfrak{p}} = \overline{R}_{\mathfrak{p}}$, and $R_{\mathfrak{p}}$ is a normal domain. Hence by Theorem 2.8 and [3, Chap. 4, §1, Proposition 5], we see that

$$\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Eass}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}/R_{\mathfrak{p}}) = \bigcup_{i=1}^{n} \mathrm{Ass}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I_{[\alpha_i]}R_{\mathfrak{p}}).$$

This implies that $\mathfrak{p} \in \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]})$ by [3, Chap. 4, §1, Proposition 5]. □

*Remark 2.10* (1) If $A$ is a finitely generated ring extension of $R$, then $\mathrm{Eass}_R(A/R)$ is a finite set by Theorem 2.9.

(2) In the notations and assumptions of Theorem 2.9, the following holds:

$$\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \subset \mathrm{Eass}_R(A/R)$$
$$\subset \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \cup \{\mathfrak{p} \in \mathrm{Dp}_1(R);\, \mathfrak{p} \supset \mathfrak{c}(\overline{R}/R)\}.$$

(3) Assume that $\overline{R}$ is a finite $R$-module. Then by the following Lemma 2.11, it is easily verified that

$$\mathrm{Ass}_R(\overline{R}/R) = \{\mathfrak{p} \in \mathrm{Spec}\,R;\, \mathfrak{p} \text{ is a prime divisor of } \mathfrak{c}(\overline{R}/R)\}.$$

Hence by the following Lemma 2.12, we have the following:

$$\bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \subset \mathrm{Eass}_R(A/R) \subset \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \cup \mathrm{Ass}_R(\overline{R}/R).$$

**Lemma 2.11** [1, Lemma 8] *Let $R$ be a Noetherian domain with quotient field $K$ and $\overline{R}$ the integral closure of $R$ in $K$. Assume that $\overline{R}$ is a finite $R$-module. Let $\mathfrak{p}$ be an element of $\mathrm{Dp}_1(R)$. If $\mathfrak{p} \supset \mathfrak{c}(\overline{R}/R)$, then $\mathfrak{p}$ is a prime divisor of $\mathfrak{c}(\overline{R}/R)$.*

**Lemma 2.12** *Let $R$ be a Noetherian domain with quotient field $K$ and $\overline{R}$ the integral closure of $R$ in $K$. Assume that $\overline{R}$ is a finite $R$-module. Let $\mathfrak{p}$ be an element of $\mathrm{Spec}\,R$. Then, the following conditions are equivalent:*

(i) *$\mathfrak{p}$ is a prime divisor of $\mathfrak{c}(\overline{R}/R)$.*
(ii) *$\mathfrak{p} \in \mathrm{Dp}_1(R)$ and $\mathfrak{p} \supset \mathfrak{c}(\overline{R}/R)$.*

*Proof* (i) $\Longrightarrow$ (ii). Let $\mathfrak{p}$ be a prime divisor of $\mathfrak{c}(\overline{R}/R)$. By [7, Proposition 1.10], we see that $\mathfrak{p} \in \mathrm{Dp}_1(R)$. Clearly, $\mathfrak{p} \supset \mathfrak{c}(\overline{R}/R)$.

(ii) $\Longrightarrow$ (i) is immediate from Lemma 2.11. □

Even if $\overline{R}$ is a finite $R$-module, the equality

$$\mathrm{Eass}_R(A/R) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\alpha_i]}) \cup \mathrm{Ass}_R(\overline{R}/R)$$

does not hold as the following example shows:

*Example 2.13* Let $k$ be a field whose characteristic is not 2. Let $k[X, Y]$ be a polynomial ring in two indeterminates $X$ and $Y$ over $k$ and set

$$R = k[X, Y]/(Y^2 - X(X-1)^2(X-2)^2) = k[x, y].$$

Then, the following seven assertions hold:

(1) $\dfrac{y}{(x-1)(x-2)}$ is integral over $R$.
(2) $\overline{R} = k[\dfrac{y}{(x-1)(x-2)}] = R[\dfrac{y}{(x-1)(x-2)}]$ and $\overline{R}$ is a finite $R$-module.
(3) Let $\mathfrak{p}_1$ be the prime ideal of $R$ generated by $x-1$ and $y$ and let $\mathfrak{p}_2$ be the prime ideal of $R$ generated by $x-2$ and $y$. Then, $\mathfrak{c}(\overline{R}/R) = \mathfrak{p}_1 \cap \mathfrak{p}_2$.
(4) Set $\varphi_\alpha(T) = T^2 + \dfrac{y}{x}T + \dfrac{1}{x-1}$. Then, $\varphi_\alpha(T)$ is irreducible over $K$ where $K$ is the quotient field of $R$.
(5) Let $\alpha$ be a root of $\varphi_\alpha(T) = 0$. Then, we have $I_{[\alpha]} = (x, y) \cap (x-1)$. Let $\mathfrak{p}_0$ be the prime ideal generated by $x$ and $y$. Then, the prime divisors of $I_{[\alpha]}$ are $\mathfrak{p}_0$ and $\mathfrak{p}_1$.
(6) Let $g(T)$ be an element of $R[T]$ and set $\beta = g(\alpha)$. Then, $I_{[\beta]} \not\subset \mathfrak{p}_2$ and $\mathfrak{p}_2$ is not a prime divisor of $I_{[\beta]}$.
(7) $\mathrm{Eass}_R(A/R) \neq \mathrm{Ass}_R(R/I_{[\alpha]}) \cup \mathrm{Ass}_R(\overline{R}/R)$.

*Proof* Set $t = \dfrac{y}{(x-1)(x-2)}$.

(1) Since $t^2 - x = 0$, $t$ is integral over $R$.

(2) First, we will show that $k[t] = R[t]$. The inclusion $k[t] \subset R[t]$ is obvious.
To prove the converse inclusion, it suffices to show that $x, y \in k[t]$. Since $x = t^2$ and $y = t(x-1)(x-2)$, we see that $x, y$ are in $k[t]$. This implies that $k[t] = R[t]$.
We note that $\overline{R} \supset k[t] \supset R$. Next, we will prove that $k[t]$ is normal. Easy calculation shows that $t$ is transcendental over $k$. Hence $k[t]$ is normal. Therefore, $\overline{R} = k[t]$.
We know that $\overline{R}$ is a finite $R$-module because $\overline{R} = R + Rt$.

(3) First, we shall show that

$$(x-1, y) \cap (x-2, y) = ((x-1)(x-2), y).$$

Since $y^2 = x(x-1)^2(x-2)^2$, every element of $R$ is of the form $a_0(x)+a_1(x)y$ $(a_0(x), a_1(x) \in k[x])$. Let $f$ be an arbitrary element of $(x-1, y)\cap(x-2, y)$. Then, we can write $f = a_0(x)+a_1(x)y$ $(a_0(x), a_1(x) \in k[x])$. Since $f$ is in $(x-1, y)$, there exist elements $g_1$, $g_2$ and $h$ of $k[X, Y]$ such that

$$f(X, Y)Y - g_1(X, Y)(X-1) - g_2(X, Y)Y = h(X, Y)(Y^2 - X(X-1)^2(X-2)^2).$$

Set $X = 1$ and $Y = 0$, then we get $f(1, 0) = a_0(1) = 0$. Similarly $a_0(2) = 0$ because $f$ is in $(x-2, y)$. Hence there exists an element $b(x)$ in $k[x]$ such that $a_0(x) = (x-1)(x-2)b(x)$. This shows that $f$ is in $((x-1)(x-2), y)$. The converse inclusion is obvious. Therefore, we have $(x-1, y) \cap (x-2, y) = ((x-1)(x-2), y)$.
Next, we shall show that $\mathfrak{c}(\overline{R}/R) = ((x-1)(x-2), y)$. The inclusion $((x-1)(x-2), y) \subset \mathfrak{c}(\overline{R}/R)$ is easily checked. We will prove the converse inclusion. Let $f$ be an arbitrary element of $\mathfrak{c}(\overline{R}/R)$. Then, we can write $f = a_0(x) + a_1(x)y$ $(a_0(x), a_1(x) \in k[x])$. Since $f\overline{R} \subset R$, there exists an element $g$ of $R$ such that $ft = g$. Hence there exists an element $h$ of $k[X, Y]$ such that

$$f(X, Y)Y - g(X, Y)(X-1)(X-2) = h(X, Y)(Y^2 - X(X-1)^2(X-2)^2).$$

This shows that $(f - hY)Y = (g - hX(X-1)(X-2))(X-1)(X-2)$. Hence there exists an element $w$ of $k[X, Y]$ such that $g - hX(X-1)(X-2) = wY$. This implies that $f - hY = w(X-1)(X-2)$. Substituting 1 and 0 for $X$ and $Y$, respectively, we have $f(1, 0) = 0$. Similarly $f(2, 0) = 0$. Therefore, $a_0(1) = 0$ and $a_0(2) = 0$, and $f$ is in $((x-1)(x-2), y)$. This claims that $\mathfrak{c}(\overline{R}/R) \subset ((x-1)(x-2), y)$. Hence $\mathfrak{c}(\overline{R}/R) = ((x-1)(x-2), y)$.
This means that $\mathfrak{c}(\overline{R}/R) = \mathfrak{p}_1 \cap \mathfrak{p}_2$.

(4) Suppose that $\varphi_\alpha(T)$ is decomposed into two linear factors over $K$. Then, there exists an element $\gamma$ of $K$ such that $\varphi_\alpha(\gamma) = 0$. Since $(x(x-1)\gamma)^2 + (x-1)y(x(x-1)\gamma) + x^2(x-1) = 0$, $x(x-1)\gamma$ is integral over $R$ and it is in $\overline{R} = k[t]$. Hence there exist elements $a_0, a_1, \ldots, a_n$ of $k$ and a non-negative integer $n$ such that $x(x-1)\gamma = a_0 + a_1 t + \cdots + a_n t^n$. Then, we get

$$(a_0 + a_1 t + \cdots + a_n t^n)^2 + (x-1)y(a_0 + a_1 t + \cdots + a_n t^n) + x^2(x-1) = 0.$$

Multiplying $(x(x-1))^{2n}$ on the both sides of the equality above, we have

$$\begin{aligned}&(a_0(x(x-1))^n + a_1 y(x(x-1))^{n-1} + \cdots + a_n y^n)^2\\ &\quad + x^n(x-1)^{n+1} y(a_0(x(x-1))^n + a_1 y(x(x-1))^{n-1} + \cdots + a_n y^n)\\ &\quad + x^{2n+2}(x-1)^{2n+1} = 0.\end{aligned}$$

Then, there exists an element $h$ of $k[X, Y]$ such that

$$\begin{aligned}&(a_0(X(X-1))^n + a_1 Y(X(X-1))^{n-1} + \cdots + a_n Y^n)^2\\ &\quad + X^n(X-1)^{n+1} Y(a_0(X(X-1))^n + a_1 Y(X(X-1))^{n-1} + \cdots + a_n Y^n)\\ &\quad + X^{2n+2}(X-1)^{2n+1} = h(X, Y)(Y^2 - X(X-1)^2(X-2)^2).\end{aligned}$$

Set $Y = 0$, then

$$a_0^2(X(X-1))^{2n} + X^{2n+2}(X-1)^{2n+1} = -h(X, 0)X(X-1)^2(X-2)^2.$$

Set $p(X) = a_0^2 + X^2(X-1)$. Then, $(X-2)^2$ divides $p(X)$. On the other hand, $p'(2) \neq 0$ where $p'(X)$ is the derivative of $p(X)$. This is a contradiction. Therefore, $\varphi_\alpha(T)$ is irreducible over $K$.

(5) The inclusion $I_{[\alpha]} \supset (x, y) \cap (x-1)$ is easily verified. We will prove the converse inclusion. Let $f$ be an arbitrary element of $I_{[\alpha]}$. Then, $f\dfrac{y}{x}$ and $f\dfrac{1}{x-1}$ are in $R$. By the condition $f\dfrac{y}{x} \in R$, there exist elements $g$ and $h$ of $k[X, Y]$ such that

$$f(X, Y)Y - Xg(X, Y) = h(X, Y)(Y^2 - X(X-1)^2(X-2)^2).$$

Hence $(f - hY)Y = X(g - h(X-1)^2(X-2)^2)$. Therefore, there exists an element $w$ of $k[X, Y]$ such that $g - h(X-1)^2(X-2)^2 = wY$. This implies that $f - hY = Xw$. Then, $f(0, 0) = 0$ and $f$ is in $(x, y)$ noting that $f$ can be written as $f = a_0(x) + a_1(x)y$ $(a_0(x), a_1(x) \in k[x])$.
Since $f\dfrac{1}{x-1}$ is in $R$, there exist elements $p$ and $q$ of $k[X, Y]$ such that

$$f(X, Y) - (X-1)p(X, Y) = q(X, Y)(Y^2 - X(X-1)^2(X-2)^2).$$

We write $f = a_0(x) + a_1(x)y,\ \ p = b_0(x) + b_1(x)y,\ \ q = c_0(x) + c_1(x)y$ $(a_0(x), a_1(x), b_0(x), b_1(x), c_0(x), c_1(x) \in k[x])$. Then, the degree with respect to $Y$ in the left side of the equality above is less than 2, we have $q(X, Y) = 0$. Therefore, $f(X, Y)$ is in $(X-1)$ and $f$ is in $(x, y) \cap (x-1)$, hence $I_{[\alpha]} = (x, y) \cap (x-1)$. Furthermore, $(X-1)$ is a $\mathfrak{p}_1$ primary ideal because $\mathfrak{p}_1$ ia a maximal ideal of $R$ and ${\mathfrak{p}_1}^2 \subset (x-1)$. Hence prime divisors of $I_{[\alpha]}$ are $\mathfrak{p}_o$ and $\mathfrak{p}_1$.

(6) Set $S = \{x^i(x-1)^j; i, j = 0, 1, 2, 3, \ldots\}$. Let $\dfrac{f}{s_0}T + \dfrac{g}{s_1}$ $(f, g \in R, s_0, s_1 \in S)$ be the remainder of $g(T)$ divided by $\varphi_\alpha(T)$. Hence $\beta = g(\alpha) = \dfrac{f}{s_0}\alpha + \dfrac{g}{s_1}$. We will find $\mu_1$ and $\mu_2$ of $K$ such that $\beta^2 + \mu_1\beta + \mu_2 = 0$. Since

$$\left(\frac{f}{s_0}\alpha + \frac{g}{s_1}\right)^2 + \mu_1\left(\frac{f}{s_0}\alpha + \frac{g}{s_1}\right) + \mu_2 = 0$$

and $\alpha^2 = -\dfrac{y}{x}\alpha - \dfrac{1}{x-1}$, we get

$$\left(-\frac{f^2y}{s_0^2x} + \frac{2fg}{s_0s_1} + \mu_1\frac{f}{s_0}\right)\alpha - \frac{f^2}{s_0^2(x-1)} + \frac{g^2}{s_1^2} + \mu_1\frac{g}{s_1} + \mu_2 = 0.$$

Hence

$$-\frac{f^2y}{s_0^2x} + \frac{2fg}{s_0s_1} + \mu_1\frac{f}{s_0} = 0,\ \ -\frac{f^2}{s_0^2(x-1)} + \frac{g^2}{s_1^2} + \mu_1\frac{g}{s_1} + \mu_2 = 0.$$

If $f = 0$, then the minimal polynomial of $\beta$ over $K$ is linear. So we may assume that $f \neq 0$. Then,

$$\mu_1 = \frac{fys_1 - 2gxs_0}{s_0s_1x},\ \ \mu_2 = \frac{f^2}{s_0^2(x-1)} - \frac{g^2}{s_1^2} - \frac{fgys_1 - 2g^2xs_0}{s_0s_1^2x}.$$

Hence the denominators of $\mu_1$ and $\mu_2$ are in $S$. If $T^2 + \mu_1T + \mu_2$ is irreducible over $K$, then $\varphi_\beta(T) = T^2 + \mu_1T + \mu_2$. If $\varphi_\beta(T) = T - \beta$, then $-\beta = -\dfrac{g}{s_1}$ and the denominator of $-\dfrac{g}{s_1}$ is in $S$. Set $h = x^m(x-1)^n$ for sufficiently large $m, n$. Then, $h$ is in $I_{[\beta]}$ and $h$ is not in $(x-2, y) = \mathfrak{p}_2$. Hence $I_{[\beta]} \not\subset \mathfrak{p}_2$ and $\mathfrak{p}_2$ is not a prime divisor of $I_{[\beta]}$.

(7) By the assertion (5), we have $\mathrm{Ass}_R(R/I_{[\alpha]}) = \{\mathfrak{p}_o, \mathfrak{p}_1\}$. The assertion (6) implies that $\mathfrak{p}_2 \notin \mathrm{Eass}_R(A/R)$. We have $\mathrm{Ass}_R(\overline{R}/R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ by the assertion (3). Hence $\mathrm{Eass}_R(A/R) \neq \mathrm{Ass}_R(R/I_{[\alpha]}) \cup \mathrm{Ass}_R(\overline{R}/R)$. □

*Remark 2.14* In Example 2.13, we know that $\mathrm{Eass}_R(A/R) = \{\mathfrak{p}_o, \mathfrak{p}_1\} = \mathrm{Ass}_R(R/I_{[\alpha]})$.

## 3 Intermediate rings

We will construct a subring of $A$ using a finite subset of $\mathrm{Eass}_R(A/R)$.

**Theorem 3.1** *Let $R$ be a Noetherian normal domain. Let $A$ be an integral domain containing $R$ such that every element of $A$ is algebraic over the quotient field of $R$. Let $\Delta$ be a finite subset of $\mathrm{Eass}_R(A/R)$. We define:*

$$A_\Delta = \{\beta \in A; \mathrm{Ass}_R(R/I_{[\beta]}) \subset \Delta\}$$

*where we define $A_\Delta = R$ if $\Delta = \emptyset$. Then, $A_\Delta$ is a subring of $A$ containing $R$.*

*Proof* If $\beta$ is in $R$, then $I_{[\beta]} = R$, and $\mathrm{Ass}_R(R/I_{[\beta]}) = \emptyset \subset \Delta$. Hence $R \subset A_\Delta$.
Let $\beta_1$ and $\beta_2$ be elements of $A_\Delta$ and set $C = R[\beta_1, \beta_2]$. Then, Theorem 2.8 implies that

$$\mathrm{Eass}_R(C/R) = \mathrm{Ass}_R(R/I_{[\beta_1]}) \cup \mathrm{Ass}_R(R/I_{[\beta_2]}) \subset \Delta.$$

Besides, $C = R[\beta_1, \beta_2, \beta_1 - \beta_2]$ and by Theorem 2.8, we get

$$\mathrm{Eass}_R(C/R) = \mathrm{Ass}_R(R/I_{[\beta_1]}) \cup \mathrm{Ass}_R(R/I_{[\beta_2]}) \cup \mathrm{Ass}_R(R/I_{[\beta_1-\beta_2]}).$$

Therefore, $\mathrm{Ass}_R(R/I_{[\beta_1-\beta_2]}) \subset \mathrm{Eass}_R(C/R) \subset \Delta$. This shows that $\beta_1 - \beta_2 \in A_\Delta$.
Similarly, using $C = R[\beta_1, \beta_2, \beta_1\beta_2]$, we have $\beta_1\beta_2 \in A_\Delta$. Hence $A_\Delta$ is a subring of $A$ containing $R$. □

We will prove that $\Delta = \mathrm{Eass}_R(A_\Delta/R)$. For this purpose we need some lemmas.

**Lemma 3.2** *Let $R$ be an integral domain and $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ non-zero primary ideals of $R$. Let $\mathfrak{p}$ be an element of* $\mathrm{Spec}R$ *with* $\mathrm{ht}\mathfrak{p} = 1$. *If $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n \subset \mathfrak{p}$, then there exists an index $i$ such that $\sqrt{\mathfrak{q}_i} = \mathfrak{p}$.*

*Proof* Since $\mathfrak{p}$ is a prime ideal of $R$, there exists an index $i$ such that $\mathfrak{q}_i \subset \mathfrak{p}$. Hence $\sqrt{\mathfrak{q}_i} \subset \mathfrak{p}$. Besides, $\sqrt{\mathfrak{q}_i}$ is a non-zero prime ideal of $R$. Therefore, $\sqrt{\mathfrak{q}_i} = \mathfrak{p}$ because of $\mathrm{ht}\mathfrak{p} = 1$. □

A primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ is called an irredundant primary decomposition if, for each index $i$, $\mathfrak{q}_i \not\supset \mathfrak{q}_1 \cap \cdots \cap \overset{\vee}{\mathfrak{q}_i} \cap \cdots \cap \mathfrak{q}_s$ where $^\vee$ denotes the symbol of deletion. We call $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ a minimal decomposition if it is an irredundant primary decomposition and $\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_s}$ are distinct.

**Lemma 3.3** *Let $R$ be a Noetherian domain with quotient field $K$ and $\alpha$ algebraic over $K$. Let $\mathfrak{p}$ be an element of* $\mathrm{Spec}R$ *such that $\mathfrak{p}$ is a prime divisor of $I_{[\alpha]}$ and* $\mathrm{ht}\mathfrak{p} = 1$. *Then, there exists an element $a$ of $R$ such that $I_{[a\alpha]}$ is a $\mathfrak{p}$-primary ideal.*

*Proof* Let $\varphi_\alpha(X)$ be the monic minimal polynomial of $\alpha$ over $K$ and write

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d, \ (\eta_1, \ldots, \eta_d \in K).$$

Let $(R :_R \eta_i) = I_i \cap \mathfrak{q}_i$ be the normal decomposition of $(R :_R \eta_i)$ where $\mathfrak{q}_i$ is a $\mathfrak{p}$-component and $I_i$ is the intersection of primary ideals except $\mathfrak{q}_i$ for $i = 1, 2, \ldots, d$. Since $\mathrm{ht}\mathfrak{p} = 1$, and $(R :_R \eta_i) = I_i \cap \mathfrak{q}_i$ is the normal decomposition of $(R :_R \eta_i)$, we get $I_i \not\subset \mathfrak{p}$ for $i = 1, 2, \ldots, d$ by Lemma 3.2. Hence $I_1 \cap \cdots \cap I_d \not\subset \mathfrak{p}$ and there exists an element $a$ of $I_1 \cap \cdots \cap I_d$ such that $a \notin \mathfrak{p}$.

We shall show that $\sqrt{(R :_R a^i \eta_i)} = \mathfrak{p}$ for $i = 1, 2, \ldots, d$. First, we will prove that $\sqrt{(R :_R a^i \eta_i)} \subset \mathfrak{p}$. Let $c$ be an arbitrary element of $(R :_R a^i \eta_i)$. Then, $ca^i \eta_i \in R$. Hence $ca^i \in (R :_R \eta_i) \subset \mathfrak{p}$. Since $a \notin \mathfrak{p}$, we have $c \in \mathfrak{p}$. Therefore, $(R :_R a^i \eta_i) \subset \mathfrak{p}$, and so $\sqrt{(R :_R a^i \eta_i)} \subset \mathfrak{p}$. Next, we will prove the converse inclusion. Let $p$ be an element of $\mathfrak{p}$. Since $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$, there exists a positive integer $n$ such that $p^n \in \mathfrak{q}_i$. Then, $p^n a \in (R :_R \eta_i)$ because $a \in I_i$. Hence $p^n a \eta_i \in R$. Therefore, $(p^n a)^i \eta_i \in R$. This shows that $p^{ni} a^i \eta_i \in R$, and $p^{ni} \in (R :_R a^i \eta_i)$. This implies that $p \in \sqrt{(R :_R a^i \eta_i)}$. Therefore, $\mathfrak{p} \subset \sqrt{(R :_R a^i \eta_i)}$.

We will prove that $(R :_R a^i \eta_i)$ is a $\mathfrak{p}$-primary ideal. Let

$$(R :_R a^i \eta_i) = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_s \cap \mathfrak{q}$$

be the normal decomposition of $(R :_R a^i \eta_i)$ and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Assume that $s \geqq 1$. Then,

$$\mathfrak{p} = \sqrt{(R :_R a^i \eta_i)} = \sqrt{\mathfrak{r}_1} \cap \cdots \cap \sqrt{\mathfrak{r}_s} \cap \mathfrak{p}.$$

Hence $\mathfrak{p} \subset \sqrt{\mathfrak{r}_1}, \ldots, \sqrt{\mathfrak{r}_s}$. Besides, $\mathfrak{p} \neq \sqrt{\mathfrak{r}_j}$ for $j = 1, \ldots, s$ because

$$(R :_R a^i \eta_i) = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_s \cap \mathfrak{q}$$

is the normal decomposition of $(R :_R a^i \eta_i)$. This means that each $\sqrt{\mathfrak{r}_j}$ is an embedded prime divisor of $(R :_R a^i \eta_i)$. This contradicts the fact that a divisorial ideal has no embedded prime divisors by [3, Chap. 7, §1, Proposition 8]. Therefore, $(R :_R a^i \eta_i) = \mathfrak{q}$ and $(R :_R a^i \eta_i)$ is a $\mathfrak{p}$-primary ideal.

Note that $I_{[a\alpha]} = \bigcap_{i=1}^{d} (R :_R a^i \eta_i)$. Hence $I_{[a\alpha]}$ is a $\mathfrak{p}$-primary ideal. □

Let $\Delta_1$ and $\Delta_2$ be finite subsets of $\mathrm{Eass}_R(A/R)$. If $\Delta_1 \subset \Delta_2$, then it is easily seen that $A_{\Delta_1} \subset A_{\Delta_2}$. Furthermore, if $\Delta_1 \neq \Delta_2$, then $A_{\Delta_1} \neq A_{\Delta_2}$ by the following theorem.

**Theorem 3.4** *Let $R$ be a Noetherian normal domain. Let $A$ be an integral domain containing $R$ such that every element of $A$ is algebraic over the quotient field of $R$. Let $\Delta$ be a finite subset of* $\mathrm{Eass}_R(A/R)$. *Then,* $\Delta = \mathrm{Eass}_R(A_\Delta/R)$.

*Proof* Let $\mathfrak{p}$ be an arbitrary element of $\mathrm{Eass}_R(A_\Delta/R)$. Then, there exists an element $\beta$ of $A_\Delta$ such that $\mathfrak{p}$ is a prime divisor of $I_{[\beta]}$. Since $I_{[\beta]}$ is a divisorial ideal of a normal domain $R$, we see that $\mathrm{ht}\mathfrak{p} = 1$ by [3, Chap. 7, §1, Theorem 3]. Then by Lemma 3.3, there exists an element $a$ of $R$ such that $I_{[a\beta]}$ is a $\mathfrak{p}$-primary ideal. On the other hand, $a\beta \in A_\Delta$ because $A_\Delta$ is a subring of $A$ containing $R$. Hence $\mathfrak{p}$ is in $\Delta$, and $\mathrm{Eass}_R(A_\Delta/R) \subset \Delta$.

Let $\mathfrak{p}$ be an element of $\Delta$. Then, there exists an element $\beta$ of $A$ such that $\mathfrak{p}$ is a prime divisor of $I_{[\beta]}$. By Lemma 3.3, there exists an element $a$ of $R$ such that $I_{[a\beta]}$ is a $\mathfrak{p}$-primary ideal. Hence $a\beta \in A_\Delta$. Furthermore, $\mathfrak{p}$ is a prime divisor of $I_{[a\beta]}$. Therefore, $\mathfrak{p}$ is in $\mathrm{Eass}_R(A_\Delta/R)$, and $\Delta \subset \mathrm{Eass}_R(A_\Delta/R)$. Hence $\Delta = \mathrm{Eass}_R(A_\Delta/R)$. □

Let $\Delta$ be a finite subset of $\mathrm{Eass}_R(A/R)$. We can construct a subring $B$ of $A$ containing $R$ such that $\Delta = \mathrm{Eass}_R(B/R)$ and $B$ is a finitely generated ring extension of $R$.

**Proposition 3.5** *Let $R$ be a Noetherian normal domain. Let $A$ be an integral domain containing $R$ such that every element of $A$ is algebraic over the quotient field of $R$. Let $\Delta$ be a finite subset of* $\mathrm{Eass}_R(A/R)$. *Then, there exists a subring $B$ of $A$ containing $R$ such that $\Delta = \mathrm{Eass}_R(B/R)$ and $B$ is a finitely generated ring extension of $R$.*

*Proof* Set $\Delta = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Note that $\mathrm{ht}\mathfrak{p}_i = 1$ for $i = 1, \ldots, n$ by [3, Chap. 7, §1, Theorem 3]. Then by Lemma 3.3, there exists an element $\beta_i$ of $A$ such that $\mathrm{Eass}_R(R[\beta_i]/R) = \mathrm{Ass}_R(R/I_{[\beta_i]}) = \{\mathfrak{p}_i\}$. Set $B = R[\beta_1, \ldots, \beta_n]$. Then, Theorem 2.8 shows that

$$\mathrm{Eass}_R(B/R) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(R/I_{[\beta_i]}) = \bigcup_{i=1}^{n} \mathrm{Eass}_R(R[\beta_i]/R) = \Delta.$$

□

*Remark 3.6* Let $R$ be a Noetherian normal domain. Let $A$ be an integral domain containing $R$ such that every element of $A$ is algebraic over the quotient field of $R$. Let $\Delta$ be a finite subset of $\mathrm{Eass}_R(A/R)$ and $B$ a subring of $A$ containing $R$ such that $\Delta = \mathrm{Eass}_R(B/R)$. Then, $B \subset A_\Delta$. Hence $A_\Delta$ is the largest subring of $A$ containing $R$ among those $B$'s such that $\Delta = \mathrm{Eass}_R(B/R)$.

*Proof* Let $\beta$ be an element of $B$. Then, every prime divisor of $I_{[\beta]}$ is in $\Delta$. Hence $\mathrm{Ass}_R(R/I_{[\beta]}) \subset \Delta$. This means that $\beta \in A_\Delta$. Therefore, $B \subset A_\Delta$. □

Finally, we pose the following question: Is $A_\Delta$ a finitely generated ring extension of $R$?

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